Numerical Methods for the Level Set Equation with Obstacles
An Application to Problems in Ecological Crime Modeling

D. J. Arnold[1], D. Fernandez[1], R. Jia[1], C. Parkinson[1], D. P. Tonne[2], Y. Yaniv[3], A. L. Bertozzi[1], S. Osher[1]

[1]University of California, Los Angeles, [2]California State University, Long Beach, [3]University of Maryland, College Park

Introduction

We consider numerical methods for an ecological crime model wherein criminals (e.g. poachers or illegal loggers) enter a protected national park area and travel inward from the boundary, while law enforcement agencies attempt to apprehend them [1]. We model the criminals' movement using the level set equation of Osher and Sethian [2]:

$$\phi_t + v\,|\nabla\phi| = 0,$$

a PDE which describes fronts propagating with normal velocity v, envisioning the front as a level set of the auxiliary function ϕ. If $v \equiv 1$, then successive level sets of ϕ correspond to curves of equal distance from the boundary. Geographic features such as lakes or mountains act as obstacles to the criminals' path inward and are accounted for by changing v.

Numerical Hamiltonians

The general Hamilton-Jacobi is given by

$$\phi_t + H(\phi_x, \phi_y) = 0.$$

The function H is called the *Hamiltonian*. Since this is fully nonlinear (in general), naïve differencing methods will not work. Instead, we replace the Hamiltonian $H(\phi_x, \phi_y)$ with a numerical Hamiltonian $\hat{H}(\phi_x^+, \phi_x^-, \phi_y^+, \phi_y^-)$ which carefully chooses which difference approximations should be used.

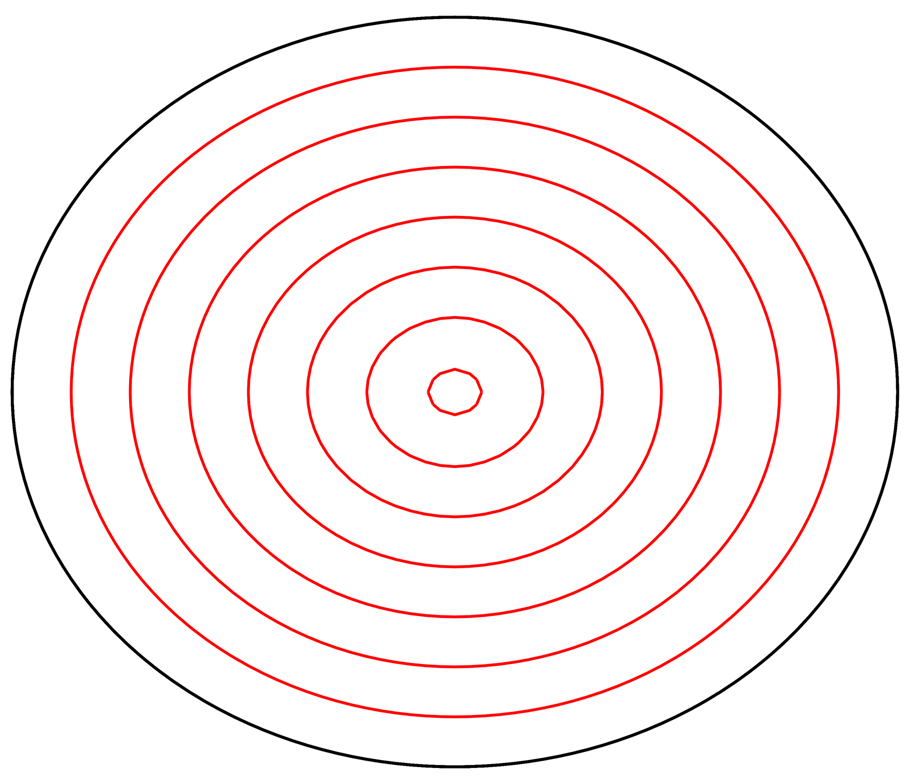

Figure 1: Circular Level Sets with Lax-Friedrichs Hamiltonian

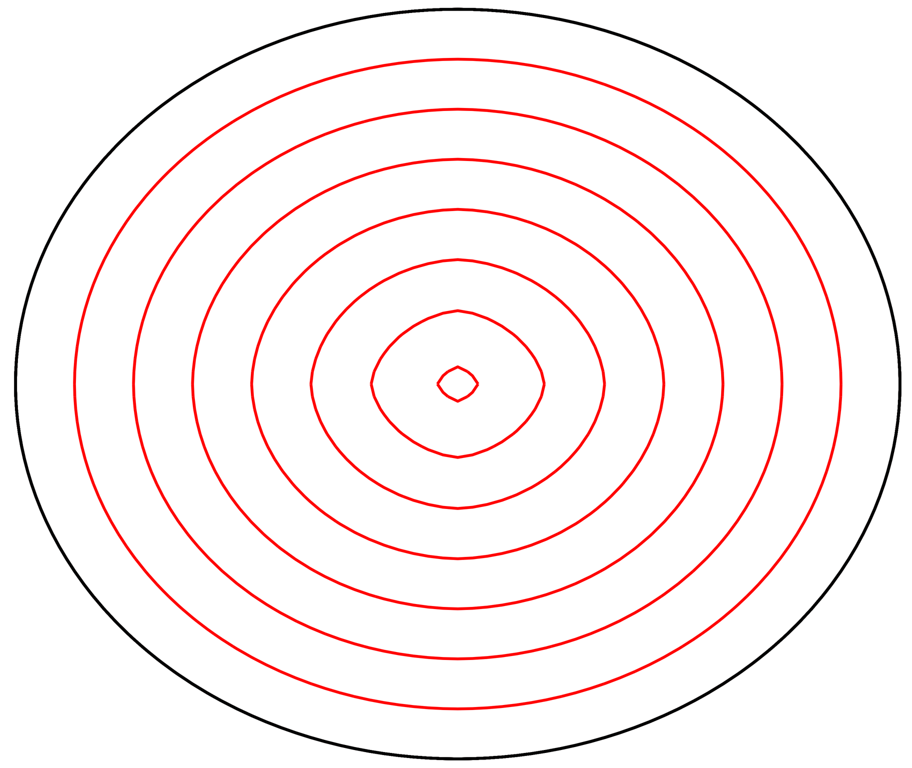

Figure 2: Circular Level Sets with Upwind Hamiltonian

We use three choices of numerical Hamiltonian (Osher & Sethian [2], Osher & Shu [3]).

(1) Lax-Friedrichs. Idea: introduce artificial diffusion to smooth solutions.

$$\hat{H}^{LF}(\phi_x^+, \phi_x^-, \phi_y^+, \phi_y^-) = H\left(\tfrac{\phi_x^+ + \phi_x^-}{2}, \tfrac{\phi_y^+ + \phi_y^-}{2}\right) - \tfrac{\alpha_1}{2}(\phi_x^+ - \phi_x^-) - \tfrac{\alpha_2}{2}(\phi_y^+ - \phi_y^-)$$

where α_1, α_2 are constants depending on H.

(2) Upwind. Idea: track the direction in which information is flowing.

$$\hat{H}^{U}(\phi_x^+, \phi_x^-, \phi_y^+, \phi_y^-) = H(\overline{\phi}_x, \overline{\phi}_y)$$

where

$$\overline{\phi}_x = \begin{cases} \sqrt{\min(\phi_x^-, 0)^2 + \max(\phi_x^+, 0)^2} & \text{if } H(\downarrow, \cdot), \\ \sqrt{\min(\phi_x^+, 0)^2 + \max(\phi_x^-, 0)^2} & \text{if } H(\uparrow, \cdot), \end{cases}$$

and similarly for $\overline{\phi}_y$.

(3) Godunov. Idea: minimize oscillation.

$$\hat{H}^{G}(\phi_x^+, \phi_x^-, \phi_y^+, \phi_y^-) = \underset{u \in I(\phi_x^+, \phi_x^-)}{\text{ext}} \; \underset{v \in I(\phi_y^+, \phi_y^-)}{\text{ext}} H(u, v)$$

where

$$I(a, b) = [\min(a, b), \max(a, b)]$$

and

$$\underset{x \in I(a,b)}{\text{ext}} = \begin{cases} \min_{a \le x \le b} & \text{if } a \le b, \\ \max_{b \le x \le a} & \text{if } a > b. \end{cases}$$

We implemented all three numerical Hamiltonians in conjunction with forward Euler time-stepping.

Circular Level Sets

Figures 1-2 show Lax-Friedrichs and Upwind level sets when the initial boundary is a circle. While both are first order accurate schemes, the Lax-Friedrichs level sets remain circular while the Upwind level sets deform into a diamond shape. This is because Lax-Friedrichs is diffusive and hence produces smooth solutions.

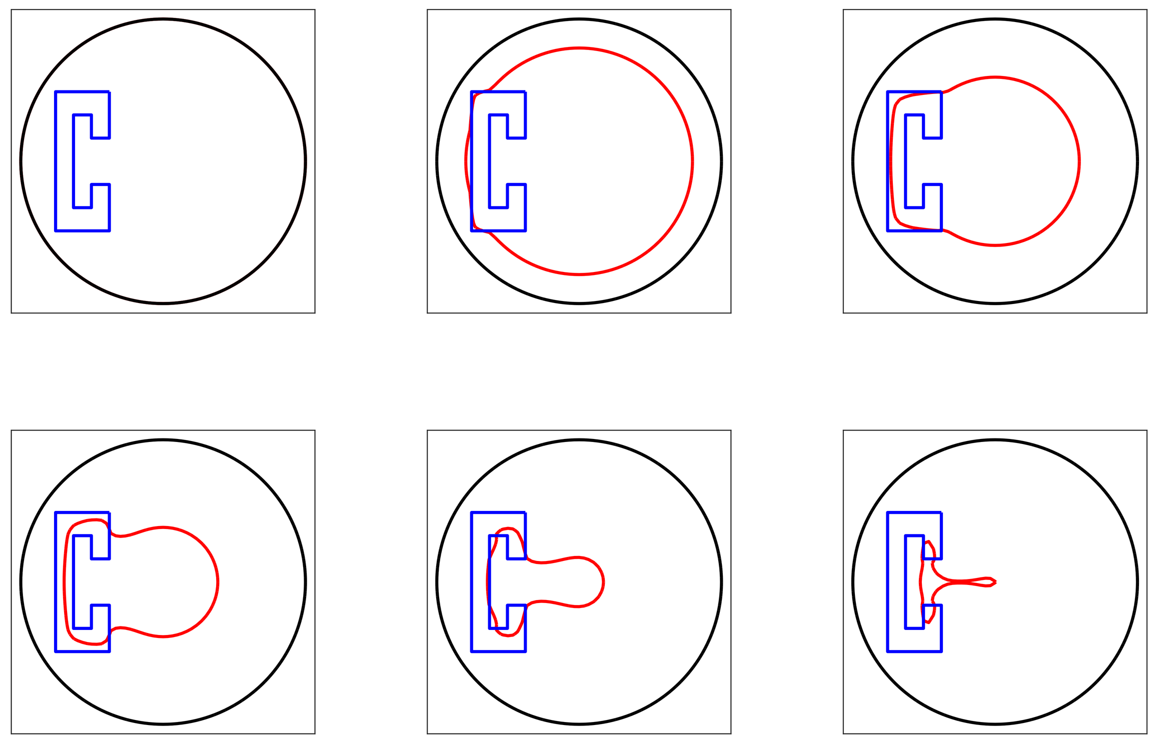

Figure 3: Lax-Friedrichs level sets diffuse through the lake

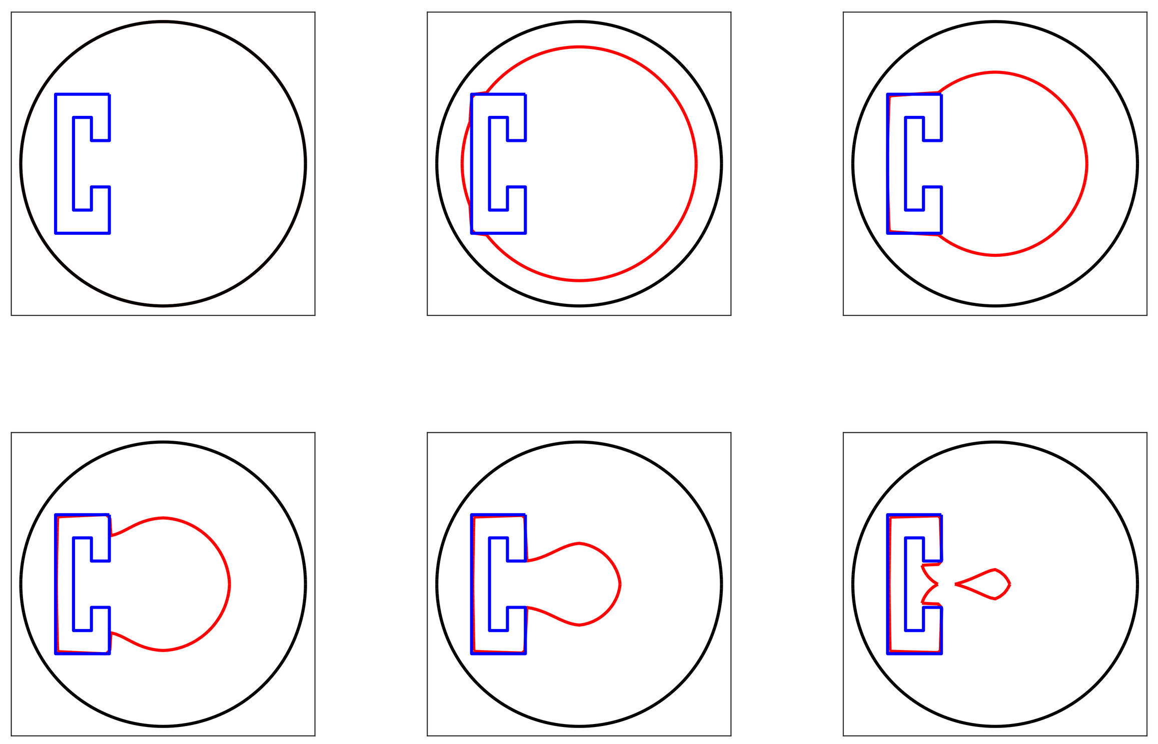

Figure 4: Godunov level sets stop at the lake

Level Sets With Obstacles

A further difference between the Hamiltonians is how they treat obstacles. We can include impassable obstacles such as lakes in our model by setting the velocity $v \equiv 0$ inside certain regions. When using the Lax-Friedrichs Hamiltonian, the level sets still move through regions where $v = 0$, such as the blue, C-shaped region in Figure 3. This is because of the artificial diffusion which is typically negligible in comparison to the Hamiltonian except when $v = 0$. The Upwind and Godunov schemes are non-diffusive so their level sets will actually stop in zero velocity regions (Figure 4).

Updating for Second Order

As a final note, one can accomplish second order accuracy by, (1) using a higher order time-stepping method (such as RK2), and (2) using better approximations to the spatial derivatives. This second point requires some care. For example, one cannot simply replace ϕ_x^+ with a centered difference approximation. Rather, to ensure stability in our numerical scheme, we must replace ϕ_x^+ with either a centered difference or a second order forward difference depending on which has lower oscillation. Taking similar precautions, one can accomplish accuracy at any degree one desires [3].

References

[1] M. Johnson, F. Fang and M. Tambe, *Patrol Strategies to Maximize Pristine Forest Area*, Proceedings of the Association for the Advancement of Artificial Intelligence (2012).

[2] S. Osher and J. Sethian, *Fronts Propagating with Curvature Dependent Speed: Algorithms Based on Hamilton-Jacobi Formulations*, Journal of Computational Physics, 79 (1988), 12-49.

[3] S. Osher and C.-W. Shu, *High Order Essentially Non-Oscillatory Schemes for Hamilton-Jacobi Equations*, SIAM Journal of Numerical Analysis, 28(4) (1991), 907–922.

Acknowledgements

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